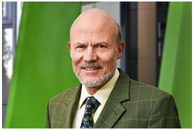
Համաչափությամբ պաշտպանված տոպոլոգիական փուլերի մոդելներ. եզրային վիճակների ֆիզիկան

Models of symmetry protected topological phases: physics of edge states

- Հեռանկարային ուղղություններով հետազոտական նախագիծ
- «Ա. Ի. Ալիխանյանի անվան ազգային գիտական լաբորատորիա» հիմնադրամ, Տեսական բաժանմունք
- ղեկավար՝ Տիգրան Հակոբյան
- ծածկագիր՝ 24FP-1F039
- տևողություն՝ 11.2024 — 11.2027

Մասնակիցներ



Անդրեաս Կլյումպեր
Վուպերտալ, Գերմանիա

Արա Սեդրակյան

Տիգրան Հակոբյան

Շահանե Խաչատրյան

Ելենա Ապրեսյան

Հրանտ Թոփչյան

Տպագրված աշխատանքներ

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Պրեպրինտներ

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Գիտաժողովներ և Սեմինարներ

- Galileo Galilei Institute, Univ. of Florence, February 2025:
  A. Sedrakyan, ...

- Student Workshop on Integrability, Budapest, June 2025:
  H. Topchyan, *Topological phases protected by* $\mathbb{Z}_3^{\times 3}$ *symmetry*

- Landau Week: Frontiers in Theoretical Physics, Yerevan, June 2025:
  - H. Topchyan, *S-matrix approach and Harris criterion in the Integer quantum Hall effect*,
  - A. Klümper, *Chiral Basis for Qubits and Spin-Helix Decay*

- Strings, Abu Dhabi, January 6-10, 2025:
  E. Apresyan, *S-move matrix in the NS sector of* $N = 1$ *super Liouville field theory*

- Quantum Theory and Symmetries (QTS-13), July–August, 2025:
  T. Hakobyan, *Quadratic Dunkl algebras and symmetries of rational Calogero models*

Quantum entanglement & Topological phase

Quantum entanglement:

- Classical many-body quantum states: $|\Psi\rangle = |\uparrow\uparrow \ldots \uparrow\rangle$
- Entangled (խճճված) many-body quantum states:

$$|\Psi\rangle = |\mathrm{GHZ}\rangle \sim |\uparrow\uparrow \ldots \uparrow\rangle + |\downarrow\downarrow \ldots \downarrow\rangle$$

- Entanglement is the main resource in quantum computers.

Topological phase:

- $|\Psi\rangle$ has intrinsic topological order or long-range entanglement, if can not be smoothly and locally deformed to a classical state, like $|\uparrow\uparrow \ldots \uparrow\rangle$
- States with topological order stay robust against noise and decoherence and can be used in topological quantum computation.
- Bulk excitations are anyons = quasiparticles with fractional statistics:

$$|a, b\rangle = e^{i\alpha}|b, a\rangle, \qquad \alpha = 0, \pi : \quad \text{bosons, fermions}$$

- Ground-state degeneracy on bulk depends on topology (torus, sphere, etc.)

Matter in (intrinsic) topological phase

Topological phase:

- Zero-energy modes appear at the edges
- Ground state is characterized by a topological invariant (winding number, Chern number, ...)

Observed in:

- Fractional quantum Hall effect with bulk anyons as excitations with fractional charge & fractional statistics [Laughlin, 1983]
- Quantum spin liquids [Anderson, 1973] in frustrated magnets (i.e. with competing spin-exchange interactions on Kagome, triangular & pyrochlore lattices) [Khatua, etc, 2023, exp. review]. Topol. prop. = magnetic monopole, skyrmion number (topological charge), Wilson loops, Berry phase & Chern numbers
- Majorana fermions at edges of topological superconductors (Kitatev chain) [Bordin, etc., 2025]

Application in:

- Quantum error correction: toric code model [Kitaev, 2003]

Symmetry-protected topological phases

Classical (Landau) phase and quantum SPT phase:

- Classical phase characterized by symmetry group G
- Quantum phase at $T = 0$: the topological pattern of ground state is essential
- Quantum matter with finite gap is characterized by [Chen, Gu, Wen, 2010]:

$$\text{symmetry} + \text{topology} = \text{symmetry-protected topological (SPT) phase}$$

Characteristic properties of SPT orders:

- single Landau phase $=$ multiple SPT phases
- Distinct SPT states cannot be smoothly and locally deformed into each other without a phase transition, if the deformation preserves the symmetry G
- short-range entanglement $=$ no intrinsic topological order
- Symmetry (t'Hooft) anomaly at the boundary
- Maps to intrinsic topological phase, for example:

$$\begin{aligned} &\text{2D Ising model} &\Rightarrow\quad &\text{Kitaev's toric code model,} \\ &\text{1D spin cluster model} &\Rightarrow\quad &\text{Majorana chain} \end{aligned}$$

Examples of SPT matter

Topological insulators:

- New states of quantum matter not equivalent to conventional insulators and semiconductors.
- Insulating gap in the bulk and gapless edge states protected by time-reversal symmetry. Conduct electricity robustly on their surfaces/edges.
- Strong spin-orbit interactions are crucial for edge states
- Theoretically predicted and experimentally observed in a variety of systems:

  $2D$ materials: HgTe quantum well — Quantum spin Hall effect

  $3D$ materials: Bi_2Se_3, Bi_2Te_3 — surface Dirac fermions
- Have potential applications in quantum computing and spintronics,

Haldane chain

- Spin-1 Heisenberg chain is gapped, but has zero-energy edge states: Haldane phase [Haldane, 1983; Affleck, etc., 1987]
- Protected by rotation symmetry $G = SO(3)$.

Symmetry protected topological orders in d dimension

- They are classified by their symmetry group cohomology composed of G-invariant cocycles ω from [Chen, Gu, Liu, Wen, 2011]

$$\mathcal{H}^{1+d}(G, U(1)), \qquad e^{i\phi} \in U(1),$$

- In particular [Moore & Seiberg, 1989]:

$$\mathcal{H}^3(\mathbb{Z}_k, U(1)) = \mathbb{Z}_k, \qquad \mathcal{H}^3(\mathbb{Z}_k \times \mathbb{Z}_k \times \mathbb{Z}_k, U(1)) = \mathbb{Z}_k^{\times 7},$$

where $\mathbb{Z}_k$ is the cyclic group by $\{1, \varepsilon, \varepsilon^2, \dots \varepsilon^{k-1}\}$ with

$$\varepsilon = \exp\left(\frac{2\pi i}{k}\right), \qquad \varepsilon^k = 1.$$

- Example: $\mathcal{H}^3(\mathbb{Z}_3)$ is generated by the nontrivial cocycle

$$\omega(n_1, n_2, n_3) = \varepsilon^{n_1 n_2 n_3 (n_2 + n_3)}, \qquad n_i = 0, 1, 2$$

Let have a 2D spin (triangular) lattice model in trivial (disordered) phase with onsite symmetry G.

Specific models:

[1] noninteracting Ising model acting on $|\uparrow\rangle$, $|\downarrow\rangle$ states; $G = \mathbb{Z}_2$

[2] noninteracting Potts model acting on $|1\rangle$, $|2\rangle$, $|3\rangle$ color states; $G = \mathbb{Z}_3$

[3] multicolor Potts model on states like $|123\rangle$; $G = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

[4] similar system for $G = \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$.

Problems:

- To construct nontrivial SPT model corresponding to $\omega \in \mathcal{H}^3(G, U(1))$
- To construct the 1D system appearing at its boundary.
- To study its low-energy behavior: check gap$= 0$ condition and calculate entanglement entropy.

  Methods: density matrix renormalization group (DMRG) & exact diagonalization (ED)
- To identify 2D conformal field theory (CFT) which describes the thermodynamic (continuum) limit.

[1] For the first model, these problems are solved exactly [Levin, Gu, 2012]

[2] Considered in our work [SciPost Phys. 18, (2025)]

[3] Considered in our previous article [JHEP, (2024)]

[4] In our current work [arXiv:2512.18460 [cond-mat], to be published]

Cochains and coboundary map in finite abelian group G

Let G be an abelian group:

- The group multiplication may be represented by $+$, and $g \to n$.
- For the cyclic group $G = \mathbb{Z}_N$,

$$g = \varepsilon^n, \qquad n = 0, 1, \dots N-1, \qquad \varepsilon = e^{\frac{2\pi i}{N}}$$

- k-cochains: $\omega = e^{i\psi} \in C^k$ with $\psi = \psi(n_1, \dots, n_k)$.
- Coboundary δ is map $C^k \to C^{k+1}$ preserving the group structure (here, linearity) which obeys [Chen, Gu, Liu & Wen, 2013; Feigin & Fuchs, 1980]:

$$\begin{aligned}\delta\psi(n_1, \dots, n_{k+1}) = {} & \psi(n_2, \dots, n_{k+1}) - \psi(n_1 + n_2, n_3, \dots, n_{k+1}) \\ & + \psi(n_1, n_2 + n_3, \dots, n_{k+1}) + \dots + (-1)^k \psi(n_1, \dots, n_{k-1}, n_k + n_{k+1}) \\ & + (-1)^{k+1} \psi(n_1, \dots, n_k)\end{aligned}$$

- This operator is nilpotent: $\delta^2\psi = 0 \mod 2\pi$, or $\delta^2\omega = 1$.

Symmetry-protected topological (SPT) ground state via unitary maps

- G-symmetric SPT phases in $d = k$ are classified by elements of $H^{k+1}(G, U(1))$.
- Any nontrivial SPT state is obtained from the trivial product state by applying a unitary U from the local quantum gates (cocyles).
- For $d = 2$:

$$U = \prod_{\langle abc \rangle} \omega_3^{\epsilon_{abc}}(n_a, n_b - n_a, n_c - n_b),$$

  where the product taken over all triangles $\langle abc \rangle$, and $\epsilon = 1$ is set for the Δ-type and $\epsilon = -1$ for ∇-type triangles.
- The unitary is symmetric:

$$[U, G] = 0,$$

- The G-invariance of U follows then from the cocycle condition $\delta\omega_3 = 1$ [Yoshida, 2017].

Trivial SPT phase: 3-state Potts model

- Recall that $\varepsilon = \exp\left(\frac{2\pi i}{3}\right)$. Define $\mathbb{Z}_3$ analogs of Pauli matrices X, Z:

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad Z = \varepsilon^n, \qquad n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

obeying: $\quad ZXZ^+ = \varepsilon X, \qquad X^3 = Z^3 = 1$

- Noninteracting 3-state Potts model on 2D triangular lattice in trivial SPT phase:

$$H_0 = -\sum_{\mathbf{r} \in \text{triang. lattice}} (X_{\mathbf{r}} + X_{\mathbf{r}}^{\dagger})$$

- Ground state is unique, gapped and is product of symmetric states:

$$|\text{gs}\rangle_0 = \prod_{\mathbf{r}} |s\rangle = |ss\ldots s\rangle, \qquad |s\rangle = |0\rangle + |1\rangle + |2\rangle.$$

- H_0 has the $\mathbb{Z}_3$ global symmetry generated by $S = \prod_p X_p$.

SPT parent Hamiltonian by unitary map

- The SPT (shifted) Hamiltonian is formed by dressed onsite Pauli matrices:

$$H = UH_0U^{-1} = -\sum_p (\bar{X}_p + \bar{X}_p^\dagger),$$

- H splits into independent (mutually commuting) bulk and boundary parts:

$$H = H_{\text{bulk}} + H_\partial$$

- The symmetry of H_∂ is broken by U, but can be restored applying the symmetrization over G.

- Alternative labeling of basic states with $w_i = 0, \pm 1$:

$$|n_1 \ldots n_N\rangle \to |w_1 \ldots w_{N-1} n_N\rangle : \quad \varepsilon^{w_i} = \varepsilon^{n_i - n_{i-1}},$$

Edge Hamiltonian: winding number symmetry

- After tedious calculation, boundary Hamiltonian is expressed as:

$$H_\partial = -\sum_{p\in\partial} \bar{X}_p \delta_{(w_p+w_{p+1})(w_{p+1}-w_p+1)} + \text{Herm. conj.}$$

with modulo 3 δ-function: $\delta_n = \frac{1}{3}(1+\varepsilon^n+\varepsilon^{-n})$

- Then the winding-number operator:

$$\mathcal{W} = \frac{1}{3}\sum_{p\in\partial} w_p = \sum_{p\in\partial}(\delta_{w_p-1} - \delta_{w_p+1})$$

counts the full number of turns that ε^{n_p} makes around the unit circle as we move around the boundary ∂.

- The winding-number invariance of the edge chain:

$$[\mathcal{W}, H_\partial] = 0$$

- $\mathcal{W}$ has topological nature with integer values $w = 0, \pm 1, \pm 2, \ldots$.
- It distinguishes the non-trivial SPT phase from the trivial one.
- This symmetry can be reformulated in terms of (discrete) conserved current:

$$
\begin{aligned}
j_p^\mu &= (q_p, m_p), \\
\partial_\mu j_p^\mu &= \partial_t q_p - \nabla_p m_p = \imath [H_\partial, q_p] - (m_{p+1} - m_{p-1}) = 0, \\
q_i &= \frac{1}{3}(w_i + w_{i+1}), \\
m_p &= \ldots
\end{aligned}
$$

Boundary 't Hooft anomaly for the SPT Hamiltonian

- Consider a single endpoint [Oshikawa, 2023] and restrict the symmetry to a half-infinite interval $p \in (0, 1, \dots)$

$$\mathcal{S}_{\text{red}}(k) = \left(\prod_{p=0}^{\infty} \varepsilon^{-kn_p n_{p+1}(n_{p+1}-n_p)} \right) \cdot \left(\prod_{p=0}^{\infty} X_p^k \right)$$

- The product satisfy fusion rule:

$$\mathcal{S}_{\text{red}}(k_1)\mathcal{S}_{\text{red}}(k_2) = \omega_3(n_0, k_1, k_2)\mathcal{S}_{\text{red}}(k_1 k_2)$$

- The anomaly appears as a phase factor ω_3 in associativity condition

$$\mathcal{S}_{\text{red}}(k_1)\left[\mathcal{S}_{\text{red}}(k_2)\mathcal{S}_{\text{red}}(k_3)\right] = \omega_3(k_1, k_2, k_3)\left[\mathcal{S}_{\text{red}}(k_1)\mathcal{S}_{\text{red}}(k_2)\right]\mathcal{S}_{\text{red}}(k_3)$$

- Here, the 3-cocycle is generator of cohomology $H^3(\mathbb{Z}_3, U(1))$:

$$\omega_3(k_1, k_2, k_3) = \varepsilon^{k_1 k_2 k_3 (k_1 + k_2)}$$

Edge model: DMRG results on excitation gap

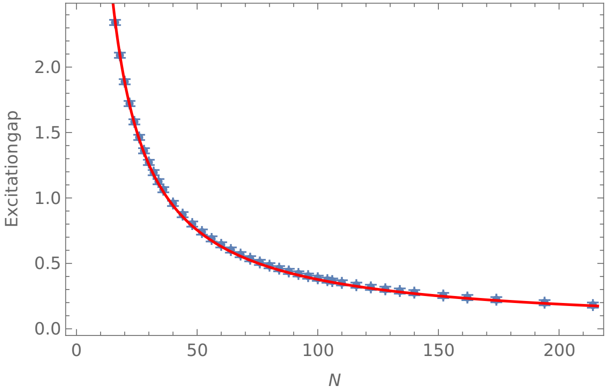

Figure: The density-matrix renormalization group (DMRG) based simulation of the first excitation gap in open chains plotted versus the boundary size up to $N = 214$. Red line is the $\propto N^{-1}$ fitting curve. The finite-size gap behavior in a CFT: $\Delta_N = \frac{2\pi}{N} x_N + \mathcal{O}(10^{-2})$ [Cardy, 1984,1986]. We found: $x_N \simeq 2.01$.

Entanglement entropy for open boundary chains: CFT description

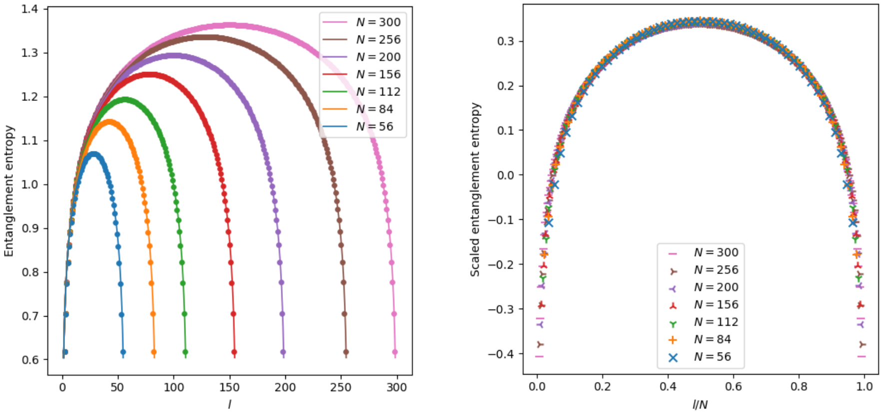

Figure: Points represent numerical values of the unscaled (left) and scaled (right) entanglement entropy (EE) for finite open chains with lengths $N = 56, \ldots, 300$.
Curves are the EE in CFT with central charge c [Calabrese & Cardy, 2009]:

$$S_N(l) = a + \frac{c}{6} \log\left(\frac{N}{\pi} \sin\left[\frac{\pi l}{N}\right]\right)$$

with $c = 1.06$ and $a = 0.557$.

Entanglement entropy for closed boundary chains: CFT description

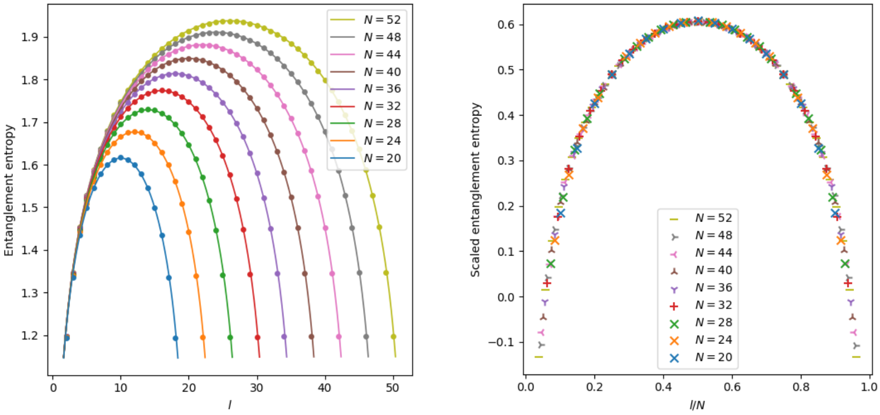

Figure: Points represent numerical values of the unscaled (left) and scaled (right) entanglement entropy for finite closed chains with lengths $N = 20, 24, ..., 52$.
Curves are the EE in CFT with central charge c [Calabrese & Cardy, 2009]:

$$S_N(l) = a + \frac{c}{3} \log\left(\frac{N}{\pi}\sin\left[\frac{\pi l}{N}\right]\right)$$

with $c = 1.01$ and $a = 0.991$.

Holomorphical factorisation into chiral $U(1)$ currents

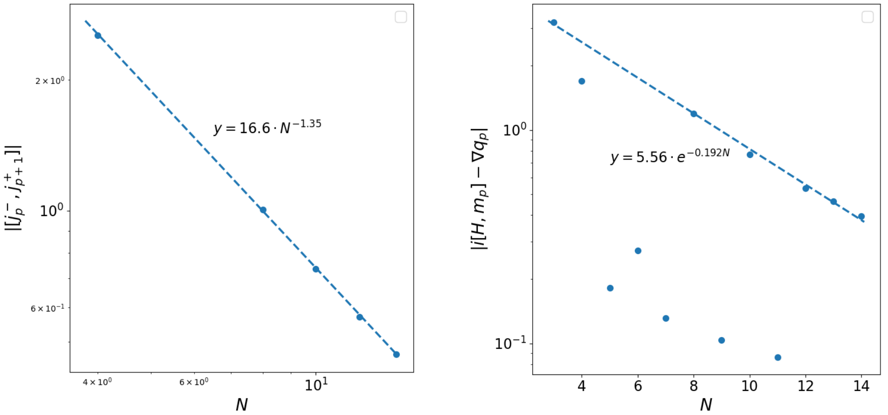

Figure: The finite-size behavior of expressions indicating the existence of anomalous $U(1)$ symmetry of the boundary Hamiltonian.

Left panel shows a power-low decrease of the commutator $[j_p^+, j_{p+1}^-]$.

Right panel shows an exponential decrease the conservation condition $i[H, m_p] - \nabla q_p$.

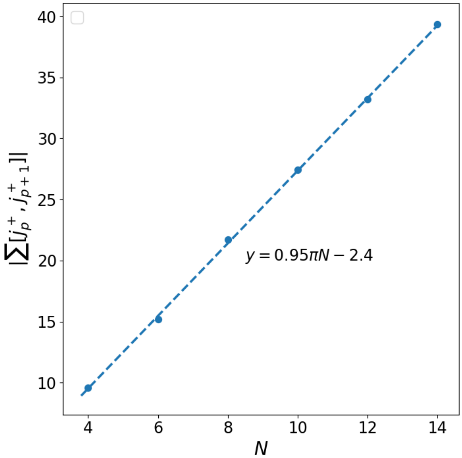

Figure: The expression

$$|\sum_p [j_p^+, j_{p+1}^+]|$$

versus system size N. The slope of the line is 0.95π, indicating that current anomaly $k \approx 1$.

Low-energy conformal field theory suggestion

- Anomaly matching rule: Anomalous behaviors of both the (edge) lattice model and the corresponding continuum field theory must be similar.
- Based on the central charge, conformal dimension of scaling operator, and level:

$$c = 1, \qquad \Delta = 2, \qquad k = 1,$$

it is conjectured that low-energy limit is described by coset conformal field theory:

$$\frac{SU_k(3)}{SU_k(2)}$$

- $SU_1(3)$ contains two $U_1(1)$ subgroups at level $k = 1$, one of which will be gauged out by the $SU_1(2)$, while the next will remain.
  It is linked to anomalous winding-number symmetry $\mathcal{W}$.
- 't Hooft anomalous $\mathbb{Z}_3$ in the boundary chain can be a subgroup of this $U_1(1)$ Kac-Moody.
- To be more precise, more studies needed in context of [Kawagoe & Levin, 2021; Else & Nayak, 2014].

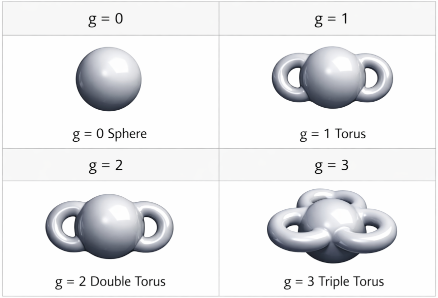

Figure: genus g=0,1,2,3 surfaces